Chapter 9. The Atomic Nucleus

Notes:

- *Most of the material in this chapter is taken from Thornton and Rex, Chapters 12 and 13.*

9.1 Nuclear Properties

Atomic nuclei are composed of *protons* and *neutrons*, which are referred to as *nucleons*. Although both types of particles are not fundamental or elementary, they can still be considered as basic constituents for the purpose of understanding the atomic nucleus. Protons and neutrons have many characteristics in common. For example, their masses are very similar with 1.0072765 u (938.272 MeV) for the proton and 1.0086649 u (939.566 MeV) for the neutron. The symbol 'u' stands for the *atomic mass unit* defined has one twelfth of the mass of the main *isotope* of carbon (i.e., $^{12}\mathrm{C}$), which is known to contain six protons and six neutrons in its nucleus. We thus have that

$$\begin{aligned} 1\ \mathrm{u} &= 1.66054 \times 10^{-27}\,\mathrm{kg} \\ &= 931.49\ \mathrm{MeV}/c^2. \end{aligned} \tag{9.1}$$

Protons and neutrons also both have the same intrinsic spin, but different magnetic moments (see below). Their main difference, however, pertains to their electrical charges: the proton, as we know, has a charge of $+e$, while the neutron has none, as its name implies.

Atomic nuclei are designated using the symbol

$$^{A}_{Z}X_{N}, \tag{9.2}$$

with Z, N and A the number of protons (atomic element number), the number of neutrons, and the atomic mass number ($A = Z + N$), respectively, while X is the chemical element symbol. It is often the case that Z and N are omitted, when there is no chance of confusion. Although the number of protons Z is fixed for a given element, the number of neutrons can vary. The different versions of an element with different N (or A) are called *isotopes*, which will arise with different frequencies. For example, carbon naturally occurs under the following forms and abundances:

$$\begin{aligned} &^{12}\mathrm{C}: \quad 98.93\% \\ &^{13}\mathrm{C}: \quad 1.07\% \\ &^{14}\mathrm{C}: \quad \text{trace}, \end{aligned} \tag{9.3}$$

where 'trace' means something like 'barely measurable.' Indeed, $^{14}\mathrm{C}$ occurs naturally with an abundance of 1 part per trillion (i.e., $1 \times 10^{-10}\%$). Furthermore, although $^{12}\mathrm{C}$ and

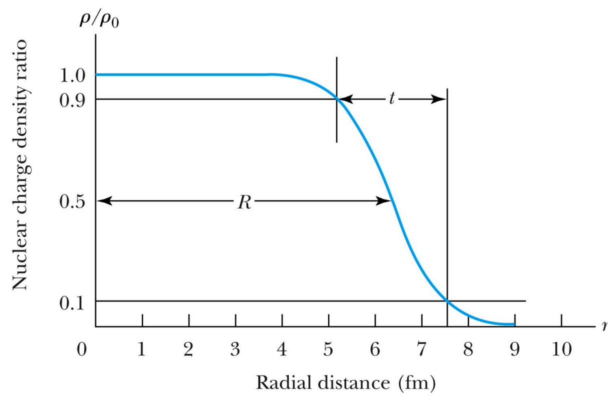

Figure 1 – The normalized nuclear charge density for an $A = 150$ nuclide.

^{13}C are stable, ^{14}C is *radioactive* (i.e., unstable) and *decays* to ^{14}N with a *half-life* of 5730 years. Finally, any nuclear species is called a *nuclide*, while those with the same number of neutrons are *isotones* (e.g., ${}^{14}_{6}C_8$ and ${}^{15}_{7}N_8$) and *isobars* have the same number of nucleons (e.g., ${}^{18}_{8}O_{10}$ and ${}^{18}_{9}F_9$).

9.1.1 Nuclear Sizes and Shapes

To a good approximation nuclei are spherical, with their volume scaling with their mass or nuclide number. Accordingly, the nuclear radius is found to be

$$R = r_0 A^{1/3}, \tag{9.4}$$

with[1] $r_0 \simeq 1.2 \times 10^{-15}\,\text{m}$. It follows that if we consider the nucleus to be a sphere of volume $V = 4\pi R^3/3$, then the average nuclear mass density is

$$\begin{aligned} \langle \rho_\text{m} \rangle &= \frac{A \cdot \text{u}}{V} \\ &= \frac{3\text{u}}{4\pi r_0^3} \\ &= 2.3 \times 10^{17}\,\text{kg/m}^3, \end{aligned} \tag{9.5}$$

which is approximately 14 orders of magnitude greater than for ordinary matter.

Except for the lightest nuclei, the charge distribution is also found to be approximately spherical with

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/a}}, \tag{9.6}$$

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[1] A *femtometer* (fm) equals $10^{-15}\,\text{m}$. Conveniently, a *fermi* (fm) is also used for that unit.

with ρ_0 the central nuclear charge density, R the radius at which the has dropped to half its central density, and $t = 4.4a$ the surface thickness measured from 90% to 10% of the central density. An example, for $A = 150$, is shown in Figure 1.

Exercises

1. What is the probability of finding a $1s^1$ electron in a hydrogen atom *inside* the nucleus (proton)?

Solution.

We know from the material covered in Chapter 7 that the $1s^1$ orbital has the wave function

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}. \tag{9.7}$$

We therefore need to calculate the following

$$\begin{aligned} P(r \le r_0) &= \int_0^{2\pi} \int_0^{\pi} \int_0^{r_0} \left|\psi_{100}(r)\right|^2 r^2 \sin(\theta)\, dr\, d\theta\, d\phi \\ &= 4\pi \int_0^{r_0} \left|\psi_{100}(r)\right|^2 r^2\, dr \\ &= \frac{4}{a_0^3} \int_0^{r_0} e^{-2r/a_0} r^2\, dr, \end{aligned} \tag{9.8}$$

which since $r_0/a_0 \approx 2 \times 10^{-6}$ is approximated by

$$\begin{aligned} P(r \le r_0) &\simeq \frac{4}{a_0^3} \int_0^{r_0} r^2\, dr \\ &\simeq \frac{4}{3}\left(\frac{r_0}{a_0}\right)^3 \\ &\approx 10^{-17}. \end{aligned} \tag{9.9}$$

Although this is an exceedingly small value, it is not zero and there is a probability that the electron will be found within the nucleus.

9.1.2 The Intrinsic Spin and Magnetic Moment

Both the neutron and proton are fermions that, like the electron, have a spin $s = 1/2$. The associated magnetic moments, however, are much weaker than that of the electron, owing to their much greater mass. As we saw in Chapter 7 the electron intrinsic magnetic moment is proportional to the Bohr magneton $\mu_{\mathrm{B}}=e\hbar/(2m_{\mathrm{e}})$, and therefore inversely proportional to its mass. For nucleons, the pertinent mass is that of the proton m_{p} (the neutron mass would also work) and the elemental magnetic moment is the *nuclear magneton*

$$\mu_{\mathrm{N}}=\frac{e\hbar}{2m_{\mathrm{p}}}. \tag{9.10}$$

Since the proton has a positive electric charge (i.e., $+e$) we would expect its intrinsic magnetic moment to have the opposite sign of that of the electron. This is indeed what is found with $\mu_{\mathrm{p}}=2.79\mu_{\mathrm{N}}$. It is somewhat surprising, however, that it does not simply scale with the ratio $-m_{\mathrm{e}}/m_{\mathrm{p}}$ since $\mu_{\mathrm{e}}=-1.00116\mu_{\mathrm{B}}$. This should serve as a warning on the limitations and dangers of using the classical model of a charge spinning about an axis through its centre to explain the intrinsic spin. This is made even more obvious when considering the intrinsic magnetic moment of the neutron $\mu_{\mathrm{n}}=-1.91\mu_{\mathrm{N}}$, which we would naively expect to be zero on the account of its electrical neutrality. This result is a reflection of the fact that the neutron is not an elementary particle, but made-up of more fundamental components (*quarks*) that do possess electrical charges. We could then expect a non-zero intrinsic magnetic moment if the charge distribution is not uniform within the neutron.

9.2 The Deuteron

The deuteron is the nucleus of an isotope of hydrogen, i.e., the deuterium $^{2}\mathrm{H}$ (or D; it has an abundance of 0.0145%, to be compared with 99.985% for $^{1}\mathrm{H}$); it is composed of a proton and a neutron. The mass of a deuterium atom is measured to be $2.014102\ \mathrm{u}$, which is slightly more than the mass of the deuteron nuclide $m_{\mathrm{d}}=2.013553\ \mathrm{u}$; the difference between the two, i.e., 0.000549 u, is basically that of the electron. On the other hand, the deuteron mass is slightly more than the sum of the proton and neutron masses. This is to be expected, as the energy that binds the proton and neutron into a deuteron must be negative (just like the energy binding the electron to the proton in the hydrogen atom is also negative at $-13.6\ \mathrm{eV}$). If we denote the binding energy by $B\left({}^{2}\mathrm{H}\right)$, then we can write

$$m_{\mathrm{d}}=m_{\mathrm{p}}+m_{\mathrm{n}}-\frac{B\left({}^{2}\mathrm{H}\right)}{c^{2}}, \tag{9.11}$$

or if we add the electron mass on both sides of equation (9.11) (neglecting the small electron binding energy)

$$M\left({}^{2}\mathrm{H}\right)=m_{\mathrm{n}}+M\left({}^{1}\mathrm{H}\right)-\frac{B\left({}^{2}\mathrm{H}\right)}{c^{2}}, \tag{9.12}$$

with $M\left({}^{A}_{Z}X_{N}\right)$ the atomic mass of ${}^{A}_{Z}X_{N}$. Equations (9.11) and (9.12) are also easily understood from the principle of conservation of energy, since, considering equation (9.11), separating the proton and neutron would require the "injection" of an energy equal to the binding energy. Inserting the masses $M\left({}^{1}\mathrm{H}\right) = 1.007825\ \mathrm{u}$, $M\left({}^{2}\mathrm{H}\right) = 2.014102\ \mathrm{u}$, and $m_{\mathrm{d}} = 2.013553\ \mathrm{u}$ in equation (9.12) yields

$$B\left({}^{2}\mathrm{H}\right) = 2.224\ \mathrm{MeV}. \tag{9.13}$$

We now readily see that the nuclear binding energy is several orders of magnitudes greater than that tying an electron to a nucleus (e.g., 13.6 eV for hydrogen). Equation (9.12) can be generalized to

$$B\left({}^{A}_{Z}X_{N}\right) = \left[Nm_{\mathrm{n}} + ZM\left({}^{1}\mathrm{H}\right) - M\left({}^{A}_{Z}X_{N}\right)\right]c^{2} \tag{9.14}$$

for the binding energy of nucleus ${}^{A}_{Z}X_{N}$, which is defined as *the energy needed to break it into free neutrons and protons*. A nucleus is said to be *stable against dissociation into free neutrons and protons* if the binding energy is positive.

We note from equation (9.13) that the amount of energy needed break a deuteron is on the same order as that of a gamma ray photon. It follows that binding energies can be determined experimentally through *photodisintegration* or *photonuclear reaction* by bombarding nuclides with incident gamma ray photons and measuring the properties of the disintegration products.

Exercises

2. Consider the scattering of gamma rays on a deuteron, which leads to its break up according to

$$\gamma + d \to p + n. \tag{9.15}$$

Using the conservations of energy and linear momentum, verify that the minimum photon energy needed for this reaction to take place is approximately equal to the binding energy of the deuteron nuclide $B_{d} = 2.224\ \mathrm{MeV}$. Assume that the speed of the proton and neutron after scattering are highly non-relativistic (i.e., $v_{\mathrm{p}}, v_{\mathrm{n}} \ll c$).

Solution.

The linear momentum of an incident photon is $\mathbf{p}$ and its energy $\hbar\omega = pc$, with $p = |\mathbf{p}|$. If we assume that $v_{\mathrm{p}}, v_{\mathrm{n}} \ll c$, then the equations for the conservation of energy and linear momentum simplify to

$$pc + m_{\mathrm{d}}c^2 = m_{\mathrm{p}}c^2 + m_{\mathrm{n}}c^2 + \frac{1}{2}m_{\mathrm{p}}v_{\mathrm{p}}^2 + \frac{1}{2}m_{\mathrm{n}}v_{\mathrm{n}}^2$$
$$\mathbf{p} = m_{\mathrm{p}}\mathbf{v}_{\mathrm{p}} + m_{\mathrm{n}}\mathbf{v}_{\mathrm{n}}. \tag{9.16}$$

We now square the last of these equations to get

$$p^2 = m_{\mathrm{p}}^2 v_{\mathrm{p}}^2 + m_{\mathrm{n}}^2 v_{\mathrm{n}}^2 + 2m_{\mathrm{p}}m_{\mathrm{n}}\mathbf{v}_{\mathrm{p}} \cdot \mathbf{v}_{\mathrm{n}}, \tag{9.17}$$

but if we are looking for the minimum photon energy, then we must find the case where the proton and neutron speeds are also minimum. Since $m_{\mathrm{p}} \simeq m_{\mathrm{n}}$ this will happen when $\mathbf{v}_{\mathrm{p}} \simeq \mathbf{v}_{\mathrm{n}}$ because then $\mathbf{v}_{\mathrm{p}} \cdot \mathbf{v}_{\mathrm{n}} \simeq v_{\mathrm{p}}^2 \simeq v_{\mathrm{n}}^2 > 0$, and therefore v_{p}^2 and v_{n}^2 are minimized for a given p^2. With these approximations for the equalities of the masses and velocities equation (9.17) can be written as

$$p^2 = 2m_{\mathrm{d}}\left(\frac{1}{2}m_{\mathrm{p}}v_{\mathrm{p}}^2 + \frac{1}{2}m_{\mathrm{n}}v_{\mathrm{n}}^2\right). \tag{9.18}$$

We also know, however, from equation (9.12) that $m_{\mathrm{d}} = m_{\mathrm{n}} + m_{\mathrm{p}} - B_{\mathrm{d}}/c^2$ or, with a slight rearrangement, $\left(m_{\mathrm{n}} + m_{\mathrm{p}}\right)c^2 = m_{\mathrm{d}}c^2 + B_{\mathrm{d}}$, which upon insertion with equation (9.18) in the first of equations (9.16) yields

$$p^2 - 2m_{\mathrm{d}}cp + 2m_{\mathrm{d}}B_{\mathrm{d}} = 0. \tag{9.19}$$

This is a simple quadratic equation that is easily solved to give, after expanding the solution in a Taylor series,

$$p \simeq \frac{B_{\mathrm{d}}}{c}\left(1 + \frac{B_{\mathrm{d}}}{2m_{\mathrm{d}}c^2}\right), \tag{9.20}$$

which when using $\hbar\omega = pc$ becomes

$$\hbar\omega_{\mathrm{min}} \simeq B_{\mathrm{d}}\left(1 + \frac{B_{\mathrm{d}}}{2m_{\mathrm{d}}c^2}\right), \tag{9.21}$$

which is approximately equal to B_{d}. This is the minimum photon energy needed for photodisintegration of the deuteron.

3. The intrinsic magnetic moment of the deuteron is experimentally determined to be $0.86\mu_{\mathrm{N}}$. Given the known corresponding values of $2.79\mu_{\mathrm{N}}$ and $-1.91\mu_{\mathrm{N}}$ for the proton and neutron, respectively, what would you deduce the spin of the deuteron to be?

Solution.

Interestingly, we note that

$$\begin{aligned} \mu_{\mathrm{p}} + \mu_{\mathrm{n}} &= 0.88\mu_{\mathrm{N}} \\ &\approx 0.86\mu_{\mathrm{N}}, \end{aligned} \tag{9.22}$$

The closeness of these two values suggests that the spins of the proton and neutron are aligned (i.e., parallel). Since these two spins are $s = 1/2$ we would expect that $s = 1$ for the deuteron; this is indeed verified experimentally.

9.3 Nuclear Forces

Scattering experiments using neutrons and protons show that the nuclear interaction acting between nucleons yields the functional forms displayed in Figure 2. Both curves have a deep negative well of width of approximately 3 fm. This negative potential implies that the nuclear force is attractive within that region. On the other hand, the strong positive reversal as we move on the left of that well at distances less than approximately 0.5 fm tells us that the force are strongly repulsive when nucleons get closer than that distance. We already know that a given nucleon has a radius and a charge distribution of about 1 fm (see equations (9.4) and (9.6)), which, when combined with the aforementioned width of the well, imply that two nucleons with centers separated by less

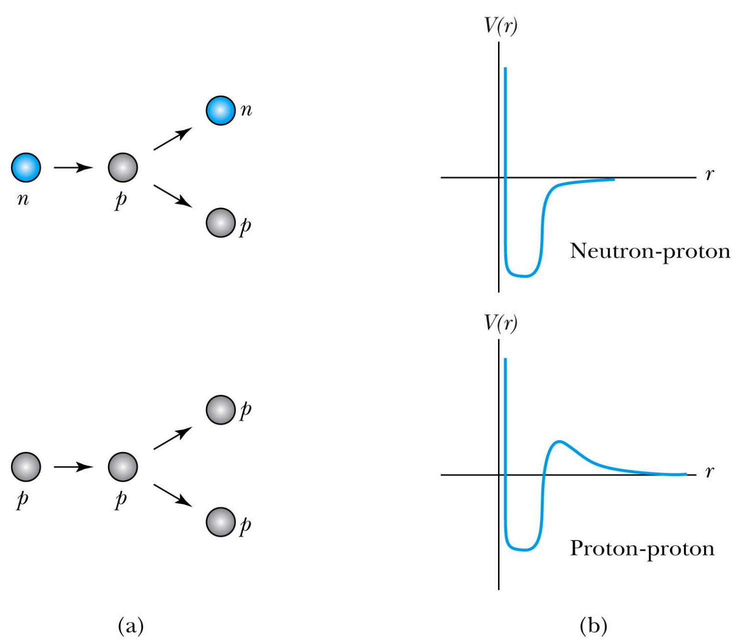

Figure 2 – The shape of the potentials measured from neutron-proton and proton-proton scattering experiments. The deep negative wells are due to the strong nuclear force, while the positive "bump" on the proton-proton potential is due to Coulomb repulsion.

than 2 or 3 fm will interact through the nuclear force. From the proton-proton potential energy curve in Figure 2 we find that the Coulomb interaction becomes important for distances greater than approximately 3 fm; this is seen from the "bump" on the right of the negative well. Otherwise, *the nuclear force is independent of electric charges*.

9.4 Nuclear Stability

We saw with equation (9.14) for the binding energy of a nuclide that it is *stable against dissociation into free neutrons and protons* if this energy is positive. But this does not guarantee that it is stable in the absolute sense. That is, stability against dissociation into free neutrons and protons does not imply that the nucleus of mass $A = Z + N$ cannot decay into another combination of two nuclei totalling the same number of nucleons. In other words, *a nucleus containing A is stable only if its mass is less than that of any other combination of A nucleons*. Mathematically this is expressed with

$$B = \left[M(R) + M(S) - M\left({}_Z^A X_N \right) \right] c^2 \tag{9.23}$$

for the binding energy of the nucleus ${}_Z^A X_N$ against dissociation into the nuclei R and S. This equation is applicable to any potential decay of a nucleus. It is important to realize, however, that it does not alone regulate whether or not a nucleus is unstable to a given dissociation. For example, the binding energy could be negative for a particular reaction that otherwise may be disallowed for other reasons (e.g., conservation of spin or other angular momenta).

Exercises

4. The main isotope of beryllium (${}_4^9\mathrm{Be}$) has an atomic mass of nine. One could wonder why an equal number of neutrons and protons, for a total mass of eight, is not allowed. First calculate the binding energy of ${}_4^8\mathrm{Be}$, and then its stability to dissociation into two distinct alpha particles (i.e., twice ${}_2^4\mathrm{He}$).

Solution.

To solve this problem we need the following nuclide masses $M\left({}_4^8\mathrm{Be}\right) = 8.005305\ \mathrm{u}$, $M\left({}_2^4\mathrm{He}\right) = 4.002603\ \mathrm{u}$, $M\left({}^1\mathrm{H}\right) = 1.007825\ \mathrm{u}$ and $m_n = 1.008665\ \mathrm{u}$. The binding energy of ${}_4^8\mathrm{Be}$ is calculated using equation (9.14)

$$\begin{aligned} B\left({}_4^8\mathrm{Be}\right) &= \left[4m_n + 4M\left({}^1\mathrm{H}\right) - M\left({}_4^8\mathrm{Be}\right) \right] c^2 \\ &= \left[4(1.008665\ \mathrm{u}) + 4(1.007825\ \mathrm{u}) - 8.005305\ \mathrm{u} \right] c^2 \left(\frac{931.5\ \mathrm{MeV}}{c^2\mathrm{u}} \right) \\ &= 56.5\ \mathrm{MeV}. \end{aligned} \tag{9.24}$$

The binding energy is positive and ${}^{8}_{4}\mathrm{Be}$ is therefore stable against break up into free neutrons and protons. But if we use equation (9.23) to find the binding of this nucleus against dissociation into two alpha particles we have

$$\begin{aligned} B\left({}^{8}_{4}\mathrm{Be}\to 2\alpha\right) &= \left[2M\left({}^{4}_{2}\mathrm{He}\right)-M\left({}^{8}_{4}\mathrm{Be}\right)\right]c^2 \\ &= \left[2\left(4.002603\ \mathrm{u}\right)-8.005305\ \mathrm{u}\right]c^2\left(\frac{931.5\ \mathrm{MeV}}{c^2\mathrm{u}}\right) \\ &= -0.093\ \mathrm{MeV}. \end{aligned} \tag{9.25}$$

This binding energy is negative and thus explains why ${}^{8}_{4}\mathrm{Be}$ is unstable and cannot be observed in nature as a stable, or the main isotope of beryllium.

5. (Prob. 61, Ch. 12 in Thornton and Rex.) Since nuclei contain in general several protons it follows that the nuclear force that binds nuclei must be able to overcome the Coulomb interaction between protons. Therefore, show that the total Coulomb self-energy of a sphere of radius *R* containing a charge *Ze* evenly distributed throughout the sphere is given by

$$\Delta E_{\mathrm{Coul}} = \frac{3}{5}\frac{\left(Ze\right)^2}{4\pi\varepsilon_0 R}. \tag{9.26}$$

Solution.

We know that interaction energy of a charge dq located at a radius r from a spherical charge distribution of total charge Q is as if it was punctual and located at $r=0$; this interaction energy is then

$$dE = \frac{Qdq}{4\pi\varepsilon_0 r}. \tag{9.27}$$

This equation sets the energy of the charge dq to zero at infinity, which we are free to choose, as a potential energy can only be specified up to an arbitrary constant. When the charge Q is a uniform sphere of radius r, then its charge density ρ is constant such that

$$Q = \frac{4}{3}\pi r^3\rho. \tag{9.28}$$

Likewise, we could consider dq to be just a small elemental charge of a spherical shell or radius r of the same density

$$dq = \rho r^2\sin\left(\theta\right)drd\theta d\phi. \tag{9.29}$$

Inserting equations (9.28) and (9.29) into equation (9.27) we have

$$dE = \frac{\rho^2}{3\varepsilon_0} r^4 \sin(\theta) dr d\theta d\phi, \tag{9.30}$$

The energy needed to bring a shell of charge dq from infinity to the surface of a sphere of similar charge density and radius r is therefore

$$\begin{aligned} dE_{\text{shell}} &= \int_0^{2\pi} \int_0^{\pi} dE \\ &= \frac{\rho^2}{3\varepsilon_0} r^4 dr \int_0^{2\pi} \int_0^{\pi} \sin(\theta) d\theta \, d\phi \\ &= \frac{4\pi}{3\varepsilon_0} \rho^2 r^4 dr. \end{aligned} \tag{9.31}$$

We can then consider the self-energy of a sphere of uniform density to be as if assembled from spherical shells of charge density ρ successively put one on top of the other. If the final sphere is of radius R, then we must integrate equation (9.31) from 0 to R. That is,

$$\begin{aligned} E_{\text{Coul}} &= \frac{4\pi}{3\varepsilon_0} \rho^2 \int_0^R r^4 \, dr \\ &= \frac{4\pi}{15\varepsilon_0} \rho^2 R^5, \end{aligned} \tag{9.32}$$

which, upon inserting equation (9.28) for the charge density while substituting $r \to R$ and $Q = Ze$, becomes

$$E_{\text{Coul}} = \frac{3}{5} \frac{(Ze)^2}{4\pi\varepsilon_0 R}. \tag{9.33}$$

Equation (9.33) shows that the energy needed to counteract the Coulomb interaction is proportional to the square of the charge of the nucleus. We also know from equation (9.4) that the radius of the nucleus scales with $A^{1/3}$, and therefore approximately with $Z^{1/3}$. We thus see that

$$E_{\text{Coul}} \propto Z^{5/3}. \tag{9.34}$$

For a given nucleus, the addition of a proton will increase the nuclear binding energy by a fixed amount, but equation (9.34) implies that the Coulomb energy will increase at a faster rate. For example, let us assume that $Z \gg 1$ and calculate the increase E_{Coul} as one goes from Z to $Z+1$ protons

$$\begin{aligned}
\Delta E_{\text{Coul}} &\propto (Z+1)^{5/3} - Z^{5/3} \\
&\propto Z^{5/3}\left[\left(1+\frac{1}{Z}\right)^{5/3} - 1\right] \\
&\propto Z^{5/3}\left[\left(1+\frac{5}{3Z}\right) - 1\right] \\
&\propto \frac{5}{3} Z^{2/3}.
\end{aligned} \tag{9.35}$$

Because the increase in the Coulomb energy gets progressively larger as the number of protons increases, and that the corresponding binding nuclear energy gain stays approximately constant, there will be a value of Z where the electrostatic repulsion will win over the nuclear force. This will then result in an unstable nuclear configuration. This is, indeed what we find in nature as the heaviest stable element is ${}^{209}_{83}\mathrm{Bi}_{126}$. Any nucleus with $Z > 83$ and $A > 209$ will decay into lighter nuclei.

Incidentally, this behaviour with increasing proton number provides a qualitative explanation for the observed tendency of heavy nuclei to have more neutrons than protons. This is one clear way to counteract the electrostatic repulsion by increasing the amount of nuclear binding energy. But the nature of nuclear interactions is not that simple. The functionality of the total binding energy of a nucleus can be well summarized by the so-called *liquid drop model* of Niels Bohr and **Carl F. von Weizsäcker** (1912-2007), which yielded the corresponding semi-empirical formula

$$B\left({}^{A}_{Z}X_{N}\right) = a_V A - a_A A^{2/3} - 0.72\ Z(Z-1)A^{-1/3} - a_S \frac{(N-Z)^2}{A} + \delta, \tag{9.36}$$

where $a_V = 15.8$ MeV, $a_A = 18.3$ MeV, $a_S = 23.2$ MeV, and

$$\delta = \begin{cases} +\Delta, & \text{for even } N \text{ and } Z \\ 0, & \text{for odd } A \text{ (even/odd } N/Z \text{ or vice-versa)} \\ -\Delta, & \text{for odd } N \text{ and } Z \end{cases} \tag{9.37}$$

with $\Delta = 33\ \mathrm{MeV} \cdot A^{-3/4}$. The first term is the *volume term* that accounts for the fact that the total binding energy is proportional to the number of nucleons (i.e. the sum of all nuclear interactions), while the second *surface effect* term reduces the binding energy because nucleons on the edge (or outer surface) of the nucleus are "missing" some nuclear interactions with non-existing neighbours. The third term is the loss in binding that is due to counteracting the Coulomb repulsion between protons. It equals the quantity calculated in equation (9.33), with the self-energy of the proton subtracted (since it is protons that we are bringing from infinity to build the nucleus; we must therefore remove the energy needed to build the protons individually), i.e.,

$$
\begin{aligned}
E_{\text{Coul}} &= \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 R}\left(Z^2 - Z\right) \\
&= \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 R} Z\left(Z-1\right).
\end{aligned}
\tag{9.38}
$$

Thus the functional form found in equation (9.36). The fourth term is the *symmetry term* and accounts for the experimental fact that nature appears to favour similar numbers of protons and neutrons as much as possible. This term is consistent with the Pauli exclusion principle, as will become apparent when considering the shell model below. Finally, the last term reflects the fact that nuclei are more stable when they possess even numbers of protons and neutrons. This is also a function of the Pauli exclusion principle since neutrons and protons are fermions, for which only pairs can occupy the same energy state (one spin up, the other spin down). Having two like nucleons with parallel spins would require for one of them to occupy a higher energy state.

It is therefore apparent from equation (9.36) that one could not simply add more neutrons in the hope that the extra binding energy would defeat the Coulomb repulsion. This is because the fourth term in the equation would also bring a reduction in binding that would eventually negate any benefit in adding neutrons.

The Bohr-von Weizsäcker liquid drop model is not the only model in existence. This plurality stems from the fact we still do not fully understand the nature of nuclear forces. Another successful interpretation of nuclear data can be obtained from the *shell model*. This model postulates a nuclear potential for the nucleons with energy states that are filled in a manner reminiscing of the atomic model discussed in Chapter 8. A schematic of the potential well is shown in Figure 3, where we see that protons are less strongly bound (with a well depth of 37 MeV) than neutrons (43 MeV) because of the electrostatic repulsion. Nuclei are then formed by filling up the energy states to minimize the total energy, with all levels below the *Fermi energy* occupied.

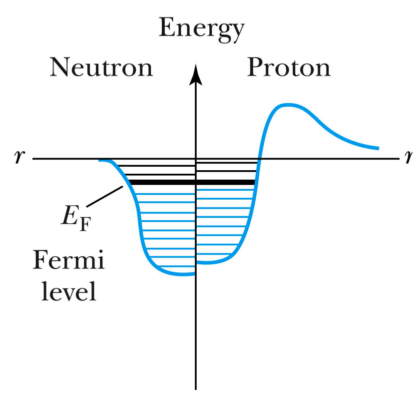

Figure 3 – Nuclear potential well for the shell model.

We can use this model (with some help from equation (9.36)) to explain, for example, the possible nuclear states between ${}^{12}_{6}\mathrm{C}$ and ${}^{16}_{8}\mathrm{O}$, as shown in Figure 4. The ${}^{12}_{6}\mathrm{C}$ nucleus, with the lowest 12 levels occupied (six each for the protons and neutrons) will be very stable on the account that $N = Z$ and $\delta = +\Delta$; the total spin is zero. The next stable nucleus with the addition of a nucleon can only be ${}^{13}_{6}\mathrm{C}_7$ as the next unoccupied level with lowest energy is the fourth neutron level; its spin will $1/2$ on the account of the unpaired neutron. It follows that ${}^{13}_{7}\mathrm{N}_6$ cannot be stable. Adding another neutron to get ${}^{14}_{6}\mathrm{C}_8$ also yields an unstable nucleus since, this time, the number of neutrons is too great for the number of protons (eight vs. six) leading to a binding energy deficit from the symmetry term in equation (9.36). Instead adding a proton leads to the stable ${}^{14}_{7}\mathrm{N}$ nucleon of spin 1 (because parallel spins have a lower energy configuration). Filling the next available energy state with a neutron gives the stable ${}^{15}_{7}\mathrm{N}_8$ isotope of spin $1/2$; ${}^{15}_{8}\mathrm{O}_7$ cannot be stable on the account of the excess Coulomb repulsion. Finally, the ${}^{16}_{8}\mathrm{O}$ nucleus (of spin zero), with the lowest 16 levels occupied will be very stable for reasons similar to ${}^{12}_{6}\mathrm{C}$.

Given the semi-empirical equation (9.36) for the binding energy, it becomes possible to investigate the stability of nuclei by dividing the binding energy by the number of nucleons, yielding the *binding energy per nucleons*. The result is shown in Figure 5. There are a few local maxima for particularly stable nuclei such as ${}^{4}_{2}\mathrm{He}$, ${}^{12}_{6}\mathrm{C}$, and ${}^{16}_{8}\mathrm{O}$, but perhaps the most important result is the fact that the binding energy per nucleon is

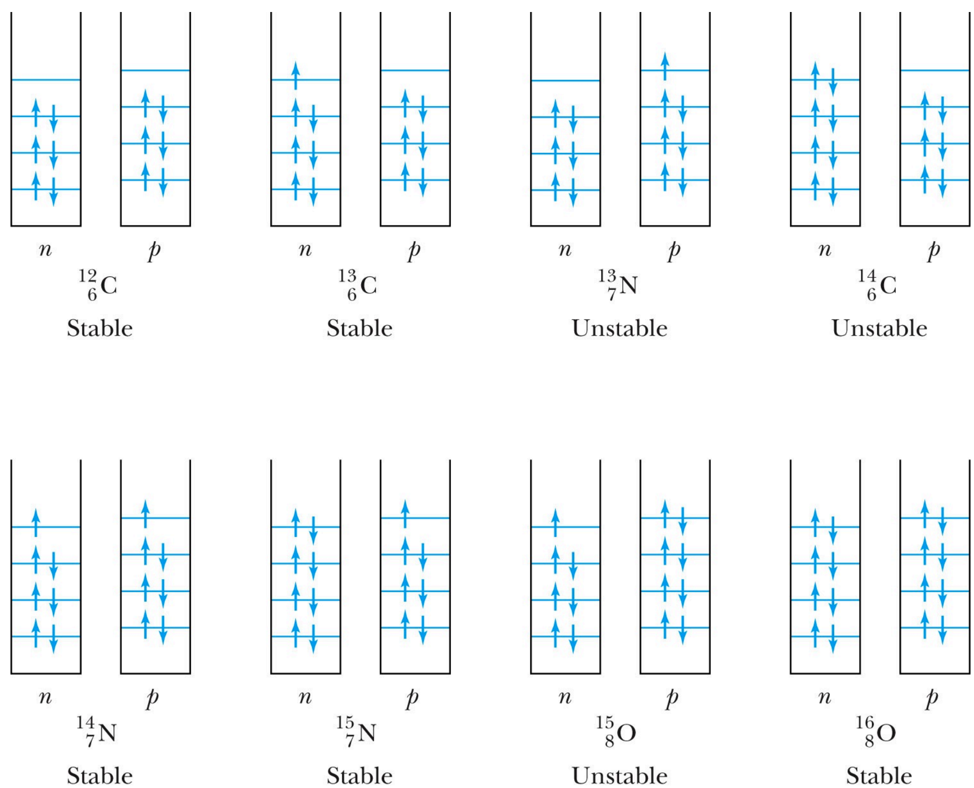

Figure 4 – Schematic of the filling of energy levels from ${}^{12}_{6}\mathrm{C}$ to ${}^{16}_{8}\mathrm{O}$.

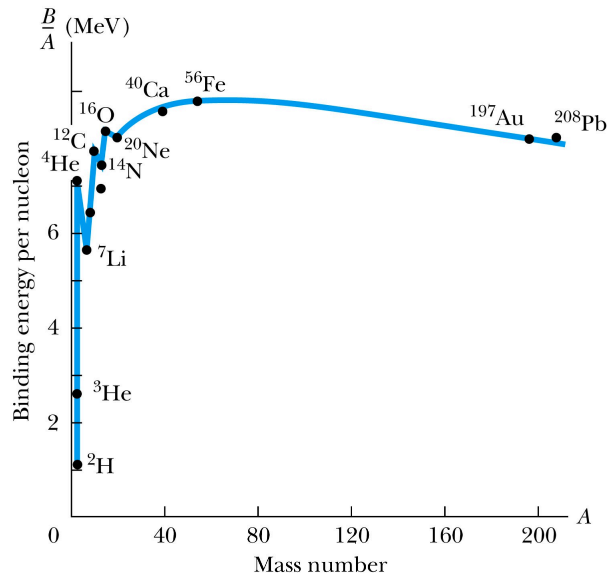

Figure 5 – The binding energy per nucleon as a function of the mass number A.

optimized at approximately $A = 56$. This is, among others, the location of ${}^{56}_{26}\mathrm{Fe}_{30}$, the main isotope of iron. It further has important consequences for the production of the different elements in the universe, and the evolution of stars.

Exercises

6. Consider the following hypothetical nuclear reaction where ${}^{56}_{26}\mathrm{Fe}_{30}$, the most stable nucleus, interacts with an element of lower atomic mass ${}^{A_1}X_1$ to produce another nuclide of higher mass ${}^{A_2}X_2$. First, show that for this reaction to be possible in general there must be another nuclide ${}^{A_3}X_3$ created. We denote the corresponding binding energy per nucleon with b_1, b_2, b_3, and b_{Fe}. We therefore have $A_1 + A_{\mathrm{Fe}} = A_2 + A_3$ (with $A_{\mathrm{Fe}} = 56$) and $b_1, b_2, b_3 < b_{\mathrm{Fe}}$. Finally, show that this creation of ${}^{A_2}X_2$ can only be exothermic if ${}^{A_1}X_1$ is of sufficiently low mass.

Solution.

The conservation of energy for a reaction that produces only ${}^{A_2}X_2$, which we assume non-relativistic, requires that

$$M\left({}^{A_1}X_1\right)c^2 + K_1 + M\left({}^{56}_{26}\mathrm{Fe}\right)c^2 + K_{\mathrm{Fe}} = M\left({}^{A_2}X_2\right)c^2 + K_2, \tag{9.39}$$

where K_i stands for the kinetic energy of nucleus i. We now define the following quantity

$$
\begin{aligned}
Q &= \left[M\left({}^{A_1}X_1 \right) + M\left({}^{56}_{26}\text{Fe} \right) - M\left({}^{A_2}X_2 \right) \right] c^2 \\
&= K_2 - \left(K_1 + K_{\text{Fe}} \right).
\end{aligned} \tag{9.40}
$$

It is clear from equation (9.40) that energy is released by the reaction when $Q > 0$ (the reaction is then said to be *exothermic*), while energy is required for the reaction to take place when $Q < 0$ (the reaction is *endothermic*). But it is also the case that linear momentum must be conserved. If we set ourselves in the reference frame where the iron nucleus is at rest (i.e., $v_{\text{Fe}} = 0$), then it is easy to show that $v_2 = v_1 m_1 / m_2$ and

$$
K_2 - K_1 = \frac{1}{2} m_1 v_1^2 \left[\frac{M\left({}^{A_1}X_1 \right)}{M\left({}^{A_2}X_2 \right)} - 1 \right]. \tag{9.41}
$$

This equation implies that such a reaction can only be exothermic if $m_1 > m_2$, which is of course impossible if only one nucleus is created as a product. As we will see later the fusion of two nuclei can often release a large amount of energy. We therefore conclude that there must be a second nucleus created at the output of this reaction, and replace equation (9.40) with

$$
\begin{aligned}
Q &= \left[M\left({}^{A_1}X_1 \right) + M\left({}^{56}_{26}\text{Fe} \right) - M\left({}^{A_2}X_2 \right) - M\left({}^{A_3}X_3 \right) \right] c^2 \\
&= K_2 + K_3 - \left(K_1 + K_{\text{Fe}} \right).
\end{aligned} \tag{9.42}
$$

We can approximately write for the total binding energies (see equation (9.14))

$$
\begin{aligned}
A_1 b_1 &\simeq \left[A_1 \text{u} - M\left({}^{A_1}X_1 \right) \right] c^2 \\
A_2 b_2 &\simeq \left[A_2 \text{u} - M\left({}^{A_2}X_2 \right) \right] c^2 \\
A_3 b_3 &\simeq \left[A_3 \text{u} - M\left({}^{A_3}X_3 \right) \right] c^2 \\
A_{\text{Fe}} b_{\text{Fe}} &\simeq \left[A_{\text{Fe}} \text{u} - M\left({}^{56}_{26}\text{Fe} \right) \right] c^2,
\end{aligned} \tag{9.43}
$$

which after insertion into equation (9.40) yields

$$
\begin{aligned}
Q &\simeq \left(A_1 + A_{\text{Fe}} - A_2 - A_3 \right) \text{u}\, c^2 - \left(A_1 b_1 + A_{\text{Fe}} b_{\text{Fe}} - A_2 b_2 - A_3 b_3 \right) \\
&\simeq \left(A_2 b_2 + A_3 b_3 \right) - \left(A_1 b_1 + A_{\text{Fe}} b_{\text{Fe}} \right),
\end{aligned} \tag{9.44}
$$

since $A_1 + A_{\text{Fe}} = A_2 + A_3$. Because we assume that $A_2 > A_{\text{Fe}}$ it must be that $A_{\text{Fe}} > A_1 > A_3$. If we now refer to Figure 5, then we find that we will most likely have $b_3 < b_1 < b_2 < b_{\text{Fe}}$.

Probably the first important thing to notice with equation (9.44) is that this reaction cannot be exothermic when $A_1 = A_{\text{Fe}}$, since we would also have $b_1 \simeq b_{\text{Fe}} > b_2, b_3$ and

$A_2 + A_3 = 2A_{\mathrm{Fe}}$. We thus find that *it is impossible to fuse two iron nuclei in an exothermic reaction to obtain a heavier nuclide*. That is, if iron is formed through nucleosynthesis in stars, then there must be another type of reaction taking place. This is because the creation of elements heavier than iron from iron only would cool down the star (i.e., $Q < 0$) and halt nuclear reactions. We can also reasonably expect a similar result when $A_1 \lesssim A_{\mathrm{Fe}}$.

Let us now rewrite equation (9.44) as follows

$$Q \simeq \left(A_2 b_2 - A_{\mathrm{Fe}} b_{\mathrm{Fe}}\right) - \left(A_1 b_1 - A_3 b_3\right) \tag{9.45}$$

and assume that $A_1 \ll A_{\mathrm{Fe}}$. We find that $A_3 < A_1$ and, referring to Figure 5, $b_3 < b_1$ when $A_2 > A_{\mathrm{Fe}}$. This implies that both terms in parentheses on the right-hand side of equation (9.45) are positive since $b_2 \lesssim b_{\mathrm{Fe}}$. It follows that the reaction will be exothermic whenever

$$A_2 b_2 - A_{\mathrm{Fe}} b_{\mathrm{Fe}} > A_1 b_1 - A_3 b_3. \tag{9.46}$$

To see if this is possible, let us write

$$\begin{aligned} A_1 &= A_3 + \Delta A, \quad & b_1 &= b_3 + \Delta b_1 \\ A_2 &= A_{\mathrm{Fe}} + \Delta A, \quad & b_2 &= b_{\mathrm{Fe}} - \Delta b_2 \end{aligned} \tag{9.47}$$

with $\Delta b_1 > \Delta b_2 > 0$ (see Figure 5), and keep calculation to first order such that equation (9.45) becomes

$$Q \simeq \Delta A\left(b_{\mathrm{Fe}} - b_3\right) - \left(A_{\mathrm{Fe}} \Delta b_2 + A_3 \Delta b_1\right). \tag{9.48}$$

Going to the limit where A_1 is so small (i.e., a few nucleons) that $\Delta b_2 \sim 0$, as well as considering that $b_{\mathrm{Fe}} \gg b_3$, we find

$$\begin{aligned} Q &\approx \Delta A\, b_{\mathrm{Fe}} - A_3 \Delta b_1 \\ &\gtrsim 0 \end{aligned} \tag{9.49}$$

whenever $b_{\mathrm{Fe}}/b_1 > A_3/\Delta A$. Although there were several approximations and assumptions leading to equation (9.49), this result nonetheless opens the possibility that, at least when based solely on conservations of energy and linear momentum considerations, it could be possible to exothermically produce elements heavier than iron by fusing it with light elements. In fact, it is known that stars can successively build heavy elements (i.e., more massive than iron) through the so-called *neutron capture process*. This is somewhat equivalent to setting ${}^{A_1}X_1 = n$ in our previous analysis, but in this process the new heavier nucleus is born into an excited state before eventually decaying to its ground state (see Section 9.5.2.4 below).

9.5 Radioactive Decay

We know from our previous discussion that nuclides that have a binding energy $B<0$ for any form of disintegration into a by-product are deemed *unstable*. There are several modes of decay, some of which will be studied in the remaining parts of this chapter. However, all modes obey the same type of decay law. First, given a sample of radioactive material the rate of decays per unit time, which we define as the *activity* R, is defined by

$$R = -\frac{dN(t)}{dt}, \tag{9.50}$$

with $N(t)$ the number of nuclides in the sample at a particular time. The SI unit for radioactive activity is the *Becquerel* ($1 \text{ Be} = 1 \text{ decay/s}$), after **Henri Becquerel** (1852-1908) an early discoverer of radioactivity with **Marie** (née **Sklodowska**, 1867-1934) and **Pierre Curie** (1859-1906), for whom another such unit exists (i.e., the *curie*, $1 \text{ Ci} = 3.7 \times 10^{4} \text{ decays/s}$).

Equation (9.50) leads to a simple decay law if the activity is proportional to the number of nuclides. More precisely, if we write

$$R = \lambda N(t), \tag{9.51}$$

with λ the *decay constant*, then we have from equation (9.50)

$$\begin{aligned} dN(t) &= -R dt \\ &= -\lambda N(t) dt. \end{aligned} \tag{9.52}$$

This first-order differential equation is readily solved as follows

$$\frac{dN(t)}{N(t)} = -\lambda dt, \tag{9.53}$$

which after integration yields

$$\ln\left[N(t)\right] = -\lambda t + C \tag{9.54}$$

or

$$N(t) = N_0 e^{-\lambda t}, \tag{9.55}$$

where N_0 is the number of nuclides at $t=0$ (it also arises in the solution from the constant of integration $C=\ln(N_0)$). Equation (9.55) is the so-called *radioactive decay law*, whose exponential nature is verified experimentally. It is often characterized by the *half-life* $t_{1/2}$, defined as the time it takes for the number of nuclides to become half the initial value (i.e., $N(t_{1/2})=N_0/2$). It is easily calculated from equation (9.55) that

$$\begin{aligned} t_{1/2} &= \frac{\ln(2)}{\lambda} \\ &\simeq \frac{0.693}{\lambda}. \end{aligned} \tag{9.56}$$

Alternatively, the *mean* or *average lifetime* τ is often used, and is determined with

$$\begin{aligned} \tau &= \frac{N_0 \int_0^\infty t e^{-\lambda t}\, dt}{N_0 \int_0^\infty e^{-\lambda t}\, dt} \\ &= \left(-\frac{t}{\lambda} e^{-\lambda t} \bigg|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t}\, dt \right) \bigg/ \int_0^\infty e^{-\lambda t}\, dt, \end{aligned} \tag{9.57}$$

or, finally,

$$\begin{aligned} \tau &= \frac{1}{\lambda} \\ &= \frac{t_{1/2}}{\ln(2)}. \end{aligned} \tag{9.58}$$

The half-lives (or lifetimes) of nuclides vary greatly, from less than 10^{-6}s to several billions years.

9.5.1 Alpha Decay

We now discuss one of the three common modes of radioactive decay, *alpha decay* (the other two are *beta* and *gamma decay*, which will follow). All of these decay process require the conservation of energy, linear and angular momenta, electric charge, and the *conservation of nucleons*. That is, this (new) law states that *the total number of nucleons* A *must be conserved in a low-energy (i.e., approximately less than the mass of a nucleon, 398 MeV) nuclear reaction or decay*.

As its name indicates, alpha decay involves the disintegration of a nucleus where one of the one of the products is an alpha particle, i.e., a $^4\mathrm{He}$ core. The radioactive nucleus, say, $^A_Z X$, is called the *parent* nuclide and two or more products can result from the disintegration process; the heaviest of the product particles is called the *daughter*. We now introduce the *disintegration energy* Q, which is the negative of the binding energy of equation (9.23) (and not unrelated to the quantity introduced in equation (9.40)), defined with

$$M\left({}_{Z}^{A}X\right)=M_D+M_y+\frac{Q}{c^2}, \tag{9.59}$$

or

$$Q=\left[M\left({}_{Z}^{A}X_N\right)-M_D-M_y\right]c^2 \tag{9.60}$$

for a disintegration into a daughter of mass M_D and one more lighter nuclide of mass M_y. We find that a nucleus is unstable for $Q>0$, i.e., the sum of the mass of the products is less than that of the parent. This is to be contrasted with the fact that a stable nucleus has $B>0$ (and therefore $Q<0$).

As mentioned above, for alpha decay the radioactive process will emit a ${}^4\text{He}$ nucleus of binding energy $B\left({}^4\text{He}\right)=28.3\text{ MeV}$. This type of reaction is written as

$${}_{Z}^{A}X\rightarrow{}_{Z-2}^{A-4}D+\alpha \tag{9.61}$$

and

$$Q=\left[M\left({}_{Z}^{A}X\right)-M\left({}_{Z-2}^{A-4}D\right)-M\left({}^4\text{He}\right)\right]c^2, \tag{9.62}$$

with $Q>0$ for the reaction to be possible. It is found that many of the nuclei with $A>150$ are susceptible to alpha decay. The high level of Coulomb interaction for these nuclides (see equation (9.36)) makes them good candidates for the ejections of nucleons, while the high stability of ${}^4\text{He}$ makes it a likely aggregate for a small number of nucleons. Radioactive decay in an alpha particle is therefore a favoured outcome.

Exercises

7. Verify if the uranium nucleus ${}_{92}^{230}\text{U}$ can alpha decay into thorium ${}_{90}^{226}\text{Th}$.

Solution.

The reaction in this case is

$${}_{92}^{230}\text{U}\rightarrow{}_{90}^{226}\text{Th}+\alpha \tag{9.63}$$

with

$$\begin{aligned} Q &= \left[M\left({}^{230}_{92}\mathrm{U}\right) - M\left({}^{226}_{90}\mathrm{Th}\right) - M\left({}^{4}\mathrm{He}\right) \right] c^2 \\ &= \left(230.033927 \text{ u} - 226.024891 \text{ u} - 4.002603 \text{ u}\right) c^2 \left(\frac{931.5 \text{ MeV}}{c^2 \text{u}} \right) \\ &= 6.0 \text{ MeV}. \end{aligned} \tag{9.64}$$

The alpha decay of ${}^{230}_{92}\mathrm{U}$ is then possible since $Q > 0$.

9.5.2 Beta Decay

Although alpha decay results from the instability of a nucleus, it is a fact that the created daughter nuclide is also often less stable than its parent. But stability can sometimes incur by simply adding or removing a charge to or from the daughter nucleus (i.e., with the following changes $Z = Z \pm 1$ and $N = N \mp 1$). If the number of nucleons A is to remain constant in the process, then this change in the charge number can be accomplished by the creation or annihilation of an electron or its *antiparticle*, the *positron*.

It was originally experimentally observed that negatively charged particles, the so-called β^- particle (now known to be electrons) were produced in some radioactive decays. However, there was a problem with the energy spectrum characterizing these electrons. For example, the decay of the unstable ${}^{14}\mathrm{C}$ isotope into the stable ${}^{14}\mathrm{N}$ should have yielded an electron with a well-defined kinetic energy, but it was generally found that the electron had significantly less energy. Furthermore, the ${}^{14}\mathrm{C}$ nucleus has a zero spin, while the main isotope of nitrogen has a spin of one. Because the electron has a spin of $1/2$ it is not possible to combine it with the spin of nitrogen to account for that of ${}^{14}\mathrm{C}$ (i.e., the combination of the nitrogen and electron spins, let us call it S, can only span the $\left|1-1/2\right| \leq S \leq \left|1+1/2\right|$ range, which does not include zero). Pauli proposed the way out of this problem in 1930 when he postulated the existence of a yet undetected particle, also of spin $1/2$. This particle, the *neutrino*, would carry the kinetic energy apparently missing by the electron and would solve the intrinsic angular momentum problem by having a spin of $1/2$. To conserve electric charge the neutrino must also be neutral. It has only recently been shown experimentally that it possesses a non-zero mass, albeit extremely small (it has yet to be precisely measured). Neutrinos are also not affected by the strong nuclear force responsible for the binding of nuclei. The beta decay is the result of the so-called *weak nuclear force* or *interaction*.

9.5.2.1 The β^- Decay

The β^- decay involves an electron and an *antineutrino*, the antiparticle of the neutrino. The simplest such decay is that of the neutron into a proton through

$$n \to p + \beta^- + \bar{\nu}, \tag{9.65}$$

where $\bar{\nu}$ is the antineutrino (ν is the neutrino). More generally, the β^- decay of the ${}^A_Z X$ nuclide is written as

$$ {}^A_Z X \to {}^A_{Z+1} D + \beta^- + \bar{\nu}, \tag{9.66} $$

with the disintegration energy

$$ Q = \left[M\left({}^A_Z X\right) - M\left({}^A_{Z+1} D\right) \right] c^2. \tag{9.67} $$

A β^- decay will only happen when $Q > 0$. For example, the aforementioned ${}^{14}\mathrm{C}$ decay

$$ {}^{14}\mathrm{C} \to {}^{14}\mathrm{N} + \beta^- + \bar{\nu} \tag{9.68} $$

yields $Q = 156.5\ \mathrm{keV}$.

9.5.2.2 The β^+ Decay

The β^+ decay process is similar to the β^- decay with the differences that it involves the creation of a positron and a neutrino. The general reaction is

$$ {}^A_Z X \to {}^A_{Z-1} D + \beta^+ + \nu, \tag{9.69} $$

where it is seen that the daughter nuclide looses a positive charge, while the disintegration energy is

$$ Q = \left[M\left({}^A_Z X\right) - M\left({}^A_{Z-1} D\right) - 2m_\mathrm{e} \right] c^2. \tag{9.70} $$

The presence of twice the electron mass is required on the right-hand side to account for the fact that equation (9.70), like equation (9.67), uses atomic masses (i.e., not nuclear masses). An example of a β^+ decay is the disintegration of ${}^{14}\mathrm{O}$ into ${}^{14}\mathrm{N}$ through

$$ {}^{14}\mathrm{O} \to {}^{14}\mathrm{N} + \beta^+ + \nu, \tag{9.71} $$

which yields $Q = 4.1\ \mathrm{MeV}$.

9.5.2.3 Electron Capture

There is another way for nuclei to loose the equivalent of a positive charge. It is actually achieved by acquiring an electron; this process is thus called *electron capture*. This reaction is mainly important for inner K- and L-shell electrons, which have a finite probability of being found in the atomic nucleus and therefore being captured by it (see Exercise 1 in this chapter). The general reaction is written as

$$ {}_{Z}^{A}X + e^{-} \rightarrow {}_{Z-1}^{A}D + \nu, \tag{9.72} $$

with

$$ Q = \left[M\left({}_{Z}^{A}X \right) - M\left({}_{Z-1}^{A}D \right) \right] c^{2}. \tag{9.73} $$

9.5.2.4 Gamma Decay

The last type of nuclear decay we consider is similar in nature to the transitions between stationary states resulting in the emission of photons by atoms. Just like atoms, nuclei can be excited to energy levels higher than their ground state. As these excited states have finite lifetimes (although they can sometimes be very long lived) they will eventually decay to lower energy states and emit gamma ray photons in the process; hence the name *gamma decay*. The gamma ray photons can have energy ranging from several keV to a few MeV.

The general decay from an excited state ${}^{A}X^{*}$ of energy $E_{>}$ (the '*' denotes an excited state) to one of lower energy $E_{<}$ is represented by

$$ {}^{A}X^{*}\left(E_{>}\right) \rightarrow {}^{A}X^{*}\left(E_{<}\right) + \gamma. \tag{9.74} $$

A decay to the ground state is simply written as

$$ {}^{A}X^{*} \rightarrow {}^{A}X + \gamma. \tag{9.75} $$

Just as atoms can be spectrally identified through the photons emitted as a result of electronic transitions, nuclei can also be identified through gamma decay photons. An

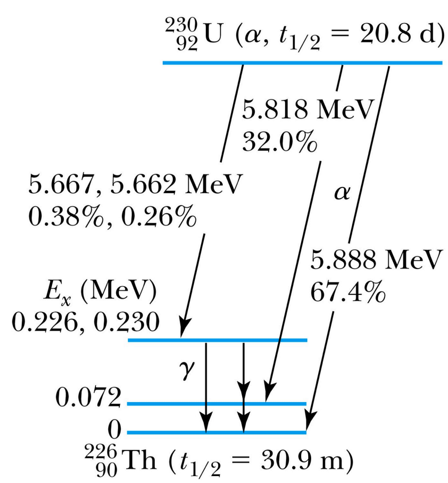

Figure 6 – Alpha and gamma decay processes for the reaction ${}^{230}_{92}\mathrm{U} \rightarrow {}^{226}_{90}\mathrm{Th}$.

example of alpha and gamma decay processes leading to the creation of the ${}^{226}_{90}\mathrm{Th}$ nuclide from ${}^{230}_{92}\mathrm{U}$ is shown in Figure 6. It is seen that the parent nuclide can alpha decay to the ground state or other low-energy excited states of the daughter nucleus. In the latter case, ${}^{226}_{90}\mathrm{Th}$ will decay to its ground state through the emission of a gamma ray photon.

9.6 Nuclear Reactions

We have so far mainly discussed the stability or instability of a nucleus by considering its binding energy B and disintegration energy Q in relation to different hypothetical rearrangements of its nucleons into different nuclei. The only exceptions to this are Exercise 2 on photonuclear reaction, and Exercise 6 on the production of nuclides heavier than iron from its fusion with a lighter nucleus. These processes are examples of *nuclear reactions*, more broadly defined when *two nuclei interact and undergo a transmutation into a different set of final nuclides*. A nuclear reaction could, in principle, involve the interaction of more than two nuclei. However, the probability of occurrence for a collision involving more than two partners is so low that this possibility can generally safely be neglected. We should also add that changes due to nuclear reactions are, in some sense, induced or forced on a nucleus and are therefore different in nature to the radioactive decays previously discussed. That is, a radioactive decay process does not constitute a nuclear reaction.

Perhaps the first such reaction ever observed was performed in Ernest Rutherford's laboratory in 1919. This reaction consisted of the bombardment of stationary ${}^{14}_{7}\mathrm{N}$ nuclei with alpha particles resulting in the following product

$$\alpha + {}^{14}_{7}\mathrm{N} \to p + {}^{17}_{8}\mathrm{O}. \tag{9.76}$$

It is generally understood that the first and second particles on the left are, respectively, the projectile and target, while on the right the detected particle is listed first and the residual nucleus last. Alternatively, equation (9.76) and nuclear reaction in general can be written using a more compact notation as follows

$${}^{14}_{7}\mathrm{N}\left(\alpha, p\right){}^{17}_{8}\mathrm{O}. \tag{9.77}$$

Going back to the discussion found in Exercise 6 we can calculate

$$\begin{aligned} Q &= \left[M\left({}^{4}\mathrm{He}\right) + M\left({}^{14}_{7}\mathrm{N}\right) - m_p - M\left({}^{15}_{8}\mathrm{O}\right)\right]c^2 \\ &= \left(4.002603\ \mathrm{u} + 14.003074\ \mathrm{u} - 1.007825\ \mathrm{u} - 16.999132\ \mathrm{u}\right)c^2\left(\frac{931.5\ \mathrm{MeV}}{c^2\mathrm{u}}\right) \\ &= -1.2\ \mathrm{MeV}. \end{aligned} \tag{9.78}$$

Since Q is also equal to the difference between the kinetic energy of the products and that of the projectile and target (see equation (9.40)), we then find that this reaction can only happen because energy was expanded in the process.

9.6.1 Fission

Nuclear fission is a process into which a nucleus separates into two fission fragments. Although *spontaneous fission*, where a heavy nucleus ($Z^2/A \geq 49$) spontaneously separate into two fission fragments without being aided by its interaction with a projectile, can in principal occur in nature, it is found that the corresponding lifetimes are several orders of magnitude greater than those associated to alpha decay.

More important to our discussion is *induced fission*, where a heavy nucleus splits after interacting with a projectile. The following is an example of such a reaction

$$n + {}^{235}_{92}\mathrm{U} \rightarrow {}^{99}_{40}\mathrm{Zr} + {}^{134}_{52}\mathrm{Te} + 3n. \tag{9.79}$$

It is generally the case that one of the two fission fragments is significantly heavier than the other, in a manner consistent with the present case. If we once again calculate the quantity Q we find

$$\begin{aligned} Q &= \left[m_n + M\left({}^{235}_{92}\mathrm{U}\right) - M\left({}^{99}_{40}\mathrm{Zr}\right) - M\left({}^{134}_{52}\mathrm{Te}\right) - 3m_n \right] c^2 \\ &= \left(235.0439\ \mathrm{u} - 98.9165\ \mathrm{u} - 133.9115\ \mathrm{u} - 2 \cdot 1.008665\ \mathrm{u}\right) c^2 \left(\frac{931.5\ \mathrm{MeV}}{c^2 \mathrm{u}} \right) \\ &= 185\ \mathrm{MeV}. \end{aligned} \tag{9.80}$$

The positive value obtained for Q implies that energy is liberated, or gained, by this nuclear reaction. In other words, referring to the result obtained in Exercise 6, although ${}^{235}_{92}\mathrm{U}$ is heavier than ${}^{56}_{26}\mathrm{Fe}$ and could therefore not be combined with a projectile to form a heavier nucleus without inputting energy in the reaction, it can fission and produce energy in the process.

Finally, the following three facts can also be observed in this example of nuclear fission. First, the amount of energy generated per nucleon (or by mass) is on the order of 1 MeV, or, more precisely, $\approx 185\ \mathrm{MeV}/236 \simeq 0.8\ \mathrm{MeV}$ per nucleon. This is an important amount of energy, which we will eventually compare with that resulting from the nuclear fusion process to be discussed below. Second, the reaction necessitates one neutron to be initiated but also liberates three neutrons as a product. This paves the way to the establishing of a *chain reaction* of fission events, which can eventually lead to a catastrophic release of energy (the atomic bomb dropped on Hiroshima at the end of the World War II used, indeed, ${}^{235}_{92}\mathrm{U}$ as nuclear fuel). Finally, one of the fragments, i.e., ${}^{134}_{52}\mathrm{Te}$, is unstable and will radioactively β^- decay. That is, the nuclear fission of ${}^{235}_{92}\mathrm{U}$

leaves behind radioactive products (or waste), which can liberate significant amounts of energy and be harmful to human when exposed to it, even after the reaction has ceased.

9.6.2 Fusion

The nuclear fission process discussed above could be described as the destruction of a heavy target nucleus into lighter fragments nuclei. The inverse process is that of *nuclear fusion*, where lighter nuclei are combined to yield a heavier one. The Rutherford reaction of equation (9.76) can be considered as such a fusion process, but we see, however, that energy is needed to create the less naturally abundant, stable ${}^{17}_{8}\mathrm{O}$ nucleus from the highly stable ${}^{14}_{7}\mathrm{N}$. But not all fusion reactions are endothermic. In fact, the following basic reaction using two isotopes of hydrogen

$$ {}^{2}\mathrm{H}+{}^{3}\mathrm{H}\to n+{}^{4}\mathrm{He} \tag{9.81}$$

produces a significant amount of energy with $Q=17.6\ \mathrm{MeV}$. This represents a release of approximately 3.5 MeV per nucleon, more than four times that attained through nuclear fission (see equation (9.80) and the discussion that follows). This higher energy throughput is characteristic of the fusion process when compared to fission. This is traced to the fact that the binding energy per nucleon increases much more drastically as we go from lighter nuclei to $A=56$ on the left part of the curve in Figure 5, than from the heavy elements on the right to $A=56$. That is, a lot more energy (per unit mass) is gained by merging light nuclei than by separating heavy nuclei.

Besides the lightest nuclei, which are believed to have form shortly after the Big Bang (e.g., hydrogen and helium), most heavy elements (no heavier than iron) are formed in stars through the fusion process. The most basic reaction is the so-called *proton-proton chain* that accounts for the creation of ${}^{4}\mathrm{He}$ as follows

$$\begin{aligned} {}^{1}\mathrm{H}+{}^{1}\mathrm{H} &\to {}^{2}\mathrm{H}+\beta^{+}+\nu \\ {}^{2}\mathrm{H}+{}^{1}\mathrm{H} &\to {}^{3}\mathrm{He}+\gamma \\ {}^{3}\mathrm{He}+{}^{3}\mathrm{He} &\to {}^{4}\mathrm{He}+{}^{1}\mathrm{H}+{}^{1}\mathrm{H}. \end{aligned} \tag{9.82}$$

This series of fusion uses a net number of four protons to form an alpha particle, while generating 26.2 MeV. Three alpha particles can eventually be successively fused to create carbon (i.e., ${}^{12}\mathrm{C}$), and open the way to the *CNO cycle*

$$
\begin{aligned}
{}^{1}\mathrm{H} + {}^{12}\mathrm{C} &\rightarrow {}^{13}\mathrm{N} + \gamma \\
{}^{13}\mathrm{N} &\rightarrow {}^{13}\mathrm{C} + \beta^{+} + \nu \\
{}^{1}\mathrm{H} + {}^{13}\mathrm{C} &\rightarrow {}^{14}\mathrm{N} + \gamma \\
{}^{1}\mathrm{H} + {}^{14}\mathrm{N} &\rightarrow {}^{15}\mathrm{O} + \gamma \\
{}^{15}\mathrm{O} &\rightarrow {}^{15}\mathrm{N} + \beta^{+} + \nu \\
{}^{1}\mathrm{H} + {}^{15}\mathrm{N} &\rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}.
\end{aligned}
\tag{9.83}
$$

It is thus seen that ${}^{12}\mathrm{C}$ merely serves as a catalyst to transmute four protons into one alpha particle, and generate 26.7 MeV in the process. The American-Austrian physicist **Hans Bethe** (1906-2005) convincingly showed that the proton-proton chain and the CNO cycle are responsible for the generation of energy in stars.